Department of Mathematics
University of Louisiana at Lafayette

Numerical Analysis Comprehensive Examination

Tuesday, August 11, 2009, 9:00AM to 1:00PM, MDD 208

This exam is closed book, but calculators are permitted (and will be useful). Please use your own paper, and put you name on each sheet.

1. Suppose f has a continuous fourth derivative. Show that

$$\left|\frac{f(x+h)-2f(x)+f(x-h)}{h^2} - f''(x)\right| = \mathcal{O}(h^2).$$

2. Assume that x^* and y^* are approximations to x and y with relative errors r_x and r_y, respectively, and that $|r_x|$, $|r_y| < R$. Assume further that $x \neq y$. How small must R be in order to ensure that $x^* \neq y^*$?

3. Consider the function $g(x) = e^{-x}$.
   (a) Show that $g(x)$ has an unique fixed point $z \in (-\infty, \infty)$.
   (b) Prove that g is a contraction on $[\ln 1.1, \ln 3]$.
   (c) Prove that $g : [\ln 1.1, \ln 3] \to [\ln 1.1, \ln 3]$.
   (d) Prove that $x_{k+1} = g(x_k)$ converges to the unique fixed point $z \in (-\infty, \infty)$ for *any* $x_0 \in (-\infty, \infty)$.

4. Suppose we iterate $x^{(k+1)} = Gx^{(k)} + k$, where $k \in \mathbb{R}^2$ and

$$G = \begin{pmatrix} 1 & -\frac{1}{3} \\ 1 & -\frac{1}{6} \end{pmatrix}.$$

   (a) Would you expect this iteration to converge? Justify your answer.
   (b) Suppose $x^* = Gx^* + k$, that is, suppose x^* is a fixed point of the iteration. If your answer to part (a) is "yes," find a $c < 1$ and a norm $\|\cdot\|$ such that

$$\|x^{(k+1)} - x^*\| \leq c\|x^{(k)} - x^*\|.$$

   Give details of your calculations that justify your choice of norm and convergence factor c.

5. Suppose

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

has approximate singular value decomposition $A = U\Sigma V^T$, given by Matlab as

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U =
   -0.2148    0.8872    0.4082
   -0.5206    0.2496   -0.8165
   -0.8263   -0.3879    0.4082
S =
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     16.8481          0          0
           0     1.0684          0
           0          0     0.0000
V =
     -0.4797    -0.7767    -0.4082
     -0.5724    -0.0757     0.8165
     -0.6651     0.6253    -0.4082
```

Write the set of all least squares solutions to $Ax = (-1, 0, 1)^T$ in the form $x \approx x^{(0)} + ty$ where $x^{(0)}$ is the least squares solution of minimum norm, $t \in \mathbb{R}$ is arbitrary, and $y \in \mathbb{R}^3$. (That is, say what $x^{(0)}$ and y are.)

6. Find the least squares approximation of

$$f(x) = \begin{cases} \frac{e^x - 1}{x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

by a linear polynomial $p(x) = ax + b$, with respect to the norm

$$\|f - p\|^2 = \int_{-1}^{1} |f(x) - p(x)|^2 dx.$$

7. What would you expect the rate of convergence of the power method to be for the matrix G of Problem 4? Explain carefully.

8. Derive a 2-point Gauss formula that integrates

$$\int_{-\pi}^{\pi} f(x) \sin(x) dx$$

exactly when f is a polynomial of degree 3 or less.

9. Derive the region of absolute stability in the complex plane for Euler's method. Show all of your work.

10. Consider the system

$$\begin{aligned} x_1^2 - x_2^2 &= 0, \\ 2x_1 x_2 &= 1. \end{aligned}$$

   (a) Perform three iterations of Newton's method on this system, using starting vector $x^{(0)} = (1/2, 1/2)^T$.

   (b) Compute the norm of the Fréchet derivative for the system at $x^{(0)}$. Based on this, predict the initial reduction of error between $x^{(0)}$ and $x^{(1)}$.