An Example to Motivate Issues in Extended Interval Arithmetic

R. Baker Kearfott

August 25, 2006

Constraint propagation in various forms is a technique used to compute narrower bounds on variables, given initial wide bounds, in many practical contexts involving systems of constraints. In this context, extended interval arithmetic is important. As an example that illustrates the importance of extended arithmetic, consider the constraint system

$$\begin{aligned} x_1^2 + x_2^2 &\leq 4.25, \\ x_1 x_2 &= 1, \end{aligned}$$

where initial bounds $x_1 \in [-2, 0]$, $x_2 \in [-2, 2]$ are known. In constraint propagation, we solve one or more of the constraints for one or more of the variables, then use the bounds on the other variables to get better bounds on the variable for which we have solved. For example, we might solve the second constraint for x_1, obtaining

$$x_1 = 1/x_2.$$

Plugging in $x_2 \in [-2, 2]$ to this, we obtain

$$x_1 \in 1/[-2, 2]. \tag{1}$$

We would obtain progress in the constraint propagation if the resulting interval value for x_1 did not contain the original range of $[-2, 0]$. However, in this case, we see a problem, because the interval by which we are dividing contains zero. We go back to the basic definition of interval arithmetic:

$$\boldsymbol{x} \text{ op } \boldsymbol{y} = \{x \text{ op } y \mid x \in \boldsymbol{x} \quad \text{and} \quad y \in \boldsymbol{y}\}.$$

In our case, x op y is not defined for all $y \in \boldsymbol{y}$. However, some thought reveals that x_1 must be in the set

$$\boldsymbol{x} \text{ op } \boldsymbol{y} = \{x \text{ op } y \mid x \in \boldsymbol{x} \quad \text{and} \quad y \in \boldsymbol{y} \quad \text{and} \quad x \text{ op } y \quad \text{is defined}\}.$$

This gives

$$x_1 \in (-\infty, -1/2] \cup [1/2, \infty).$$

When we intersect this set with the original interval $[-2, 0]$, we get $x_1 \in [-2, -1/2]$, a useful reduction in the uncertainty in the value of x_1.

This extended interval arithmetic can be defined operationally, as was originally done in 1968 by William Kahan. However, it is not clear what to do with the arithmetic in various special cases, such as when one of the end points is zero or in various cases with more complicated expressions or with certain transcendental functions. This lack of clarity has led to different people devising conflicting systems. For example, how should

$$1/[0, 1]$$

be defined? Some systems define it to be $[1, \infty)$, while other systems return the useless result $(-\infty, \infty)$. The problem with saying it it $[1, \infty)$ is the way that the left end point of the denominator arose. Other questions are concerned with whether ∞ should be considered as part of the number system, or just as a way of describing an open interval without bounds. (That is, are we looking at the real numbers or at a compactification of the real numbers?)

Cset arithmetic is an attempt to place extended arithmetic on a solid theoretical foundation, to simplify thinking about it and remove questions about how operations are defined. Basically, in cset arithmetic $\boldsymbol{x}$ op $\boldsymbol{y}$ is defined to be the set of all limits of x op y from the set

$$\{x \in \boldsymbol{x} \quad y \in \boldsymbol{y}, \quad \text{and} \quad x \text{ op } y \quad \text{is defined}\}.$$

Furthermore, in cset arithmetic, $-\infty$ and ∞ are considered to be numbers. Thus, in cset arithmetic,

$$\begin{aligned} 1/[0, 1] &= [-\infty, -\infty] \cup [1, \infty], \qquad \text{while} \\ 1/[-2, 2] &= [-\infty, -1/2] \cup [1/2, \infty]. \end{aligned}$$

However, there are still questions about this system. In particular, in some contexts it may not be convenient to work with sets of intervals that have more than one component (that is, that consist of more than a simple interval); note that the smallest single interval that encloses $\{-\infty\} \cup [1, \infty]$ is the whole real line, useless in further computations. An alternate proposal is to describe intent of computations by using open and closed intervals. (Interval arithmetic systems typically have assumed closed intervals, except, perhaps, intervals that are infinite in extent.) What do you think?

These issues are important in standardization of interval arithmetic, as well as in assembly of libraries for interval arithmetic. In turn, standardization and libraries make it easier for people to use each others' work to advance the state of knowledge in the subject area and to tackle significant applications.

One project for this course is to gather information about how various packages actually handle extended interval arithmetic.