

$$\textcircled{1} P_{B \rightarrow B'} = P_{E \rightarrow B'} P_{B \rightarrow E} = (P_{B' \rightarrow E})^{-1} P_{B \rightarrow E}.$$

$$P_{B \rightarrow E} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \text{ and } P_{B' \rightarrow E} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

$$(P_{B' \rightarrow E})^{-1}: \begin{bmatrix} 1 & 2 & | & 1 & 0 \\ 2 & 1 & | & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & | & 1 & 0 \\ 0 & -3 & | & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & | & 1 & 0 \\ 0 & 1 & | & 2/3 & -1/3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & | & -1/3 & 2/3 \\ 0 & 1 & | & 2/3 & -1/3 \end{bmatrix}, \text{ so } (P_{B' \rightarrow E})^{-1} = \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix}$$

$$\text{Thus, } P_{B \rightarrow B'} = \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1/3 & 1 \\ 1/3 & -1 \end{bmatrix}} = P_{B \rightarrow B'}.$$

$\textcircled{2}$ We compute $\text{rref}(A)$:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus:

\textcircled{a} A basis for the row space is $\{[1 \ 0 \ 3 \ 2], [0 \ 1 \ 0 \ 1]\}$.

\textcircled{b} A basis for the column space is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

\textcircled{c} The null space is parametrized by:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \text{ so a basis is } \left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

\textcircled{d} The dimension of the row space is 2.

\textcircled{e} The dimension of the column space is 2.

\textcircled{f} The dimension of the null space is 2.

\textcircled{g} The rank of A is 2.

\textcircled{h} The rows of A are not linearly independent.

\textcircled{i} The columns of A are not linearly independent.

$$A = \begin{bmatrix} \cos(\pi/6) & 0 & -\sin(\pi/6) \\ 0 & 1 & 0 \\ \sin(\pi/6) & 0 & \cos(\pi/6) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \sqrt{3}/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} = \boxed{\begin{bmatrix} \sqrt{3} & 0 & -1/6 \\ 0 & 1 & 0 \\ 1 & 0 & \sqrt{3}/6 \end{bmatrix}}$$
