1 To find the row space, column space, and null space, we reduce A to row echelon form. (This was actually done for this matrix on exam 3.) We obtained

$$\begin{bmatrix} 1 & 0 & 5/3 & 2/3 \\ 0 & 1 & 2/3 & 5/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for reduced row echelon form. (Note: only REF is needed but RREF is also ok.)

From this:

(a) $x_1 = \cancel{9} - 5/3\, x_3 - \frac{2}{3} x_4$
$x_2 = -2/3\, x_3 - 5/3\, x_4$

for the set of solutions of the corresponding set of homogeneous equations.

Thus, a basis for the null space is:

$$\left\{ \begin{bmatrix} -5/3 \\ -2/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2/3 \\ -5/3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(b) ~~The~~ a basis for the column space of A is obtained from the columns of A corresponding to leading variables. Thus, a basis for the column space of A is:

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

(c) a basis for the row space of A is:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 5/3 \\ 2/3 \end{bmatrix}^T, \begin{bmatrix} 0 \\ 1 \\ 2/3 \\ 5/3 \end{bmatrix}^T \right\}.$$

(2) (a) $P_{B'\to B} = P_{S\to B'}P_{B\to S} = (P_{B'\to S})^{-1}P_{B\to S}$.

$P_{B\to S} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, and $P_{B'\to S} = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$.

We compute $(P_{B'\to S})^{-1}$:

$$\begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 1 & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 1 & 0 \\ 0 & -2/\sqrt{2} & 1 & 1 \end{bmatrix}$$

$$\sim \left[\begin{array}{cc|cc} -1/\sqrt{2} & -1/\sqrt{2} & 1 & 0 \\ 0 & 1 & -1/\sqrt{2} & -1/\sqrt{2} \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1/\sqrt{2} & -1/\sqrt{2} \end{array}\right]$$

$$\sim \begin{bmatrix} 1 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1 & -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}, \text{ so } (P_{B'\to S})^{-1} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.$$

Thus, $P_{B'\to B} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

(b) $[w]_{B'} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

(c) $w = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$