1 (a-b)

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ -1 & 1 & -1 & 1 & 0 \\ 0 & 3 & 2 & 5 & 0 \\ 2 & 1 & 4 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & 3 & 2 & 5 & 1 \\ 0 & 3 & 2 & 5 & 0 \\ 0 & -3 & -2 & -5 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & 1 & 2/3 & 5/3 & 1/3 \\ 0 & 0 & 0 & 0 & -1/3 \\ 0 & 0 & 0 & 0 & 1/3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 2/3 & 5/3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 5/3 & 2/3 & 0 \\ 0 & 1 & 2/3 & 5/3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the third equation in the RREF gives $0=1$, there are no solutions.

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2 $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

(a) A vector in the null space has $x_1 = -2x_2$, $x_3 = 0$, that is, if $x_2 = t$, $\vec{v} = t\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, so a basis for the null space is $\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right\}$.

(b) A basis for the column space is

$\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}\right\}$.

(c) A basis for the row space is

$\{[1, 2, 0], [0, 0, 1]\}$

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3 a $P_{B\to S} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$ and $P_{B'\to S} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$.

$P_{B\to B'} = (P_{B'\to S})^{-1} P_{B\to S}$. We now compute $(P_{B'\to S})^{-1}$:

$$\left[\begin{array}{cc|cc} 1/2 & -\sqrt{3}/2 & 1 & 0 \\ \sqrt{3}/2 & 1/2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -\sqrt{3} & 2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -\sqrt{3} & 2 & 0 \\ 0 & 2 & -\sqrt{3} & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & -\sqrt{3} & 2 & 0 \\ 0 & 1 & -\sqrt{3}/2 & 1/2 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1/2 & \sqrt{3}/2 \\ 0 & 1 & -\sqrt{3}/2 & 1/2 \end{array}\right]$$

Thus $P_{B\to B'} = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} = \boxed{\begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}}$

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b $w_{B'} = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} -1/2 \\ -\sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} -\sqrt{3}/2 \\ -1/2 \end{bmatrix}$

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c $[w]_S = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} -1/2 \\ -\sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

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4. $\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & 0 & +1 \\ 0 & \lambda-2 & 0 \\ +1 & 0 & \lambda-2 \end{vmatrix} = (\lambda-2)\begin{vmatrix} \lambda-2 & +1 \\ +1 & \lambda-2 \end{vmatrix}$

$= (\lambda-2)[(\lambda-2)^2 - 1] = (\lambda-2)^3 - (\lambda-2)$. $\lambda = 2$ is an eigenvalue, and any remaining eigenvalues satisfy $(\lambda-2)^2 - 1 = 0$, that is, $\lambda^2 - 4\lambda + 3 = 0$, i.e. $(\lambda-3)(\lambda-1) = 0$.

The eigenvalues are thus $\lambda = 1$, $\lambda = 2$, and $\lambda = 3$.

For $\underline{\lambda = 1}$: $\begin{bmatrix} -1 & 0 & +1 \\ 0 & -1 & 0 \\ -1 & 0 & +1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This gives $v_1 = +v_3$, $v_2 = 0$,

so a basis for the eigenspace is $\left\{\begin{bmatrix} +1 \\ 0 \\ 1 \end{bmatrix}\right\}$

For $\underline{\lambda = 2}$: $\begin{bmatrix} 0 & 0 & +1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = 0, v_3 = 0$,

so a basis for the eigenspace is

$\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$.

For $\underline{\lambda = 3}$: $\begin{bmatrix} +1 & 0 & 1 \\ 0 & +1 & 0 \\ 1 & 0 & +1 \end{bmatrix}$, giving $\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$

as a basis for the eigenspace.

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5. Since $AP = PD$, where D is diagonal and P is a matrix whose columns are linearly independent eigenvectors, a matrix P can be:

$$\boxed{P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}$$