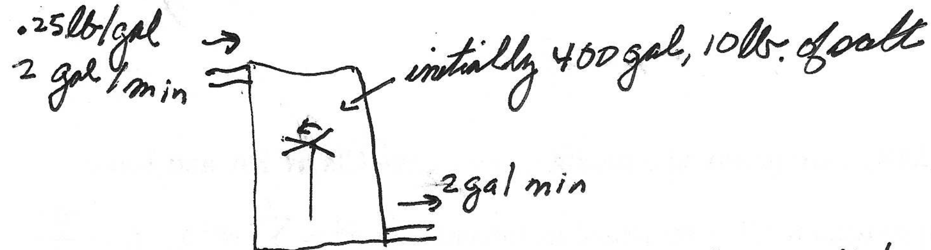
1



[Rate in] $= [(.25)\text{ lb/gal}][2\text{ gal/min}] = 0.5\text{ lb/min}$

[Rate out] $= (Q/400\text{ lb/gal})[2\text{ gal/min}] = Q/200\text{ lb/min}.$

$Q' = 0.5 - \frac{1}{200}Q, \quad Q(0) = 10.$

$Q' + \frac{1}{200}Q = 0.5. \quad \mu = e^{\int \frac{1}{200}dt} = e^{t/200}.$

$\left(e^{t/200}Q\right)' = 0.5e^{t/200} \Rightarrow e^{t/200}Q = 100e^{t/200} + C$

$\Rightarrow Q(t) = 100 + Ce^{-t/200}.$

$Q(0) = 10 = 100 + C \Rightarrow C = -90, \cancel{90}$

(a) $\boxed{Q(t) = 100 - 90e^{-t/200}}.$

(b) After a long period of time, there will be approximately 100 ~~gallons~~ pounds of salt in the tank.

2) The characteristic equation is:

$$r^2+12r+37=0 \Rightarrow r = \frac{-12}{2} \pm \frac{\sqrt{144-4(37)}}{2} = -6 \pm i.$$

Thus, $y_h = e^{-6t}\left[C_1\cos(t)+C_2\sin(t)\right]$.

a particular non-homogeneous solution is of the form:

$$y = A\cos(t)+B\sin(t) \quad \times 37: \quad 37A\cos(t) + 37B\sin(t)$$
$$y' = -A\sin(t)+B\cos(t) \quad \times 12: \; +12B\cos(t) - 12A\sin(t)$$
$$y'' = -A\cos(t)-B\sin(t) \quad \times 1: \; -A\cos(t) - B\sin(t)$$
$$\overline{(36A+12B)\cos(t) + (36B-12A)\sin(t)}$$
$$= \sin(t)$$

$$\left\{\begin{matrix} 36A+12B=0 \\ -12A+36B=1 \end{matrix}\right\} \Rightarrow B=-3A \Rightarrow -12A-108A=-120A=1$$

$$\Rightarrow A = \frac{-1}{120},\; B = \frac{3}{120} = \frac{1}{40} \Rightarrow y_p = \frac{-1}{120}\cos(t)+\frac{1}{40}\sin(t),$$

so the general solution is:

$$y_g(t) = e^{-6t}\left[C_1\cos(t)+C_2\sin(t)\right] - \frac{1}{120}\cos(t)+\frac{1}{40}\sin(t)$$

$$y(0)=1=C_1-\frac{1}{120} \Rightarrow C_1 = \frac{1}{120}+1 = \frac{121}{120}.$$

$$y'(t) = -6e^{-6t}\left[C_1\cos(t)+C_2\sin(t)\right]$$
$$+e^{-6t}\left[-C_1\sin(t)+C_2\cos(t)\right] + \frac{1}{120}\sin(t)+\frac{1}{40}\cos(t)$$

$$y'(0) = -6C_1+C_2+\frac{1}{40}=0 \Rightarrow C_2 = \frac{-1}{40}+6\left(\frac{121}{120}\right) = \frac{-1}{40}+\frac{121}{20}$$
$$= \frac{241}{40} = C_2$$

(a) $$\boxed{\therefore\; y(t) = e^{-6t}\left[\frac{121}{120}\cos(t)+\frac{241}{40}\sin(t)\right] - \frac{1}{120}\cos(t)+\frac{1}{40}\sin(t)}$$

(b) The steady-state solution is

$$y_s(t) = \frac{-1}{120}\cos(t)+\frac{1}{40}\sin(t)$$

3 $y' = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$

$(x+1)y = (x+1)\sum_{n=0}^{\infty}a_n x^n = \sum_{n=0}^{\infty}a_n x^{n+1} + \sum_{n=0}^{\infty}a_n x^n = \sum_{n=1}^{\infty}a_{n-1}x^n + \sum_{n=0}^{\infty}a_n x^n$

so $y' + (x+1)y = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty}a_n x^n + \sum_{n=1}^{\infty}a_{n-1}x^n$

$= a_1 + a_0 + \sum_{n=1}^{\infty}[(n+1)a_{n+1} + a_n + a_{n-1}]x^n$

Thus, $\boxed{a_1 + a_0 = 0}$ and, for $n \geq 1$: $(n+1)a_{n+1} + a_n + a_{n-1} = 0$

$\Rightarrow \boxed{a_{n+1} = \frac{-1}{n+1}(a_n + a_{n-1})}$.

We have: $\boxed{a_0 = 1}$, $\boxed{a_1 = -1}$, $n=1$: $a_2 = \frac{-1}{2}(-1+1) = \boxed{0 = a_2}$.

$n=2$: $a_3 = \frac{-1}{3}(a_2 + a_1) = \frac{-1}{3}(0-1) = \boxed{\frac{1}{3} = a_3}$

$n=3$: $a_4 = \frac{-1}{4}(a_3 + a_2) = -\frac{1}{4}\left(\frac{1}{3} + 0\right) = \boxed{\frac{-1}{12} = a_4}$

$n=4$: $a_5 = \frac{-1}{5}(a_4 + a_3) = \frac{-1}{5}\left(\frac{-1}{12} + \frac{1}{3}\right) = \frac{-1}{5}\left(\frac{3}{12}\right) = \frac{-1}{5}\left(\frac{1}{4}\right) = \boxed{\frac{-1}{20} = a_5}$.

Thus, the degree 5 polynomial approximation is:

$y \approx \boxed{P_5(x) = 1 - x + \frac{1}{3}x^3 - \frac{1}{12}x^4 - \frac{1}{20}x^5}$.

(4) The equation is:

$$y'' + 3y' + 2y = u_1(t) - u_2(t), \text{ so the Laplace transform is:}$$

$$s^2 Y - s y(0) - y'(0) + 3sY - 3y(0) + 2Y = \frac{1}{s}e^{-s} - \frac{1}{s}e^{-2s}.$$

so $(s^2+3s+2)Y - s - 3 = \frac{1}{s}e^{-s} - \frac{1}{s}e^{-2s} = \frac{1}{s}[e^{-s} - e^{-2s}]$, that is,

$$Y = \frac{s}{s^2+3s+2} + \frac{3}{s^2+3s+2} + \frac{1}{s(s^2+3s+2)}[e^{-s} - e^{-2s}].$$

We have 3 partial fraction decompositions to do.

$$\frac{s}{s^2+3s+2} = \frac{A}{s+1} + \frac{B}{s+2} \Rightarrow s = A(s+2) + B(s+1) = (A+B)s + (2A+B)$$

$$\Rightarrow \left.\begin{array}{l} [A+B=1] \\ 2A+B=0 \\ \boxed{A=-1, B=2} \end{array}\right\}$$

$$\frac{3}{s^2+3s+2} = \frac{C}{s+1} + \frac{D}{s+2} \Rightarrow 3 = C(s+2) + D(s+1) = (C+D)s + (2C+D)$$

$$\Rightarrow \begin{array}{l} 2C+D=3 \\ -[C+D=0] \\ \boxed{C=3, D=-3} \end{array}$$

$$\frac{1}{s(s^2+3s+2)} = \frac{E}{s} + \frac{F}{s+1} + \frac{G}{s+2} \Rightarrow 1 = E(s+1)(s+2) + F(s)(s+2) + G(s)(s+1)$$

$$= (E+F+G)s^2 + (3E+2F+G)s + (2E)$$

$$\Rightarrow 2E = 1 \Rightarrow \boxed{E = \tfrac{1}{2}} \qquad \left.\begin{array}{l} E+F+G=0 \Rightarrow [F+G=-\tfrac{1}{2}] \\ 3E+2F+G=0 \Rightarrow 2F+G=\tfrac{-3}{2} \end{array}\right\} \Rightarrow \boxed{F=-1}, \boxed{G=\tfrac{1}{2}}.$$

Thus,
$$Y(s) = \frac{-1}{s+1} + \frac{2}{s+2} + \frac{3}{s+1} - \frac{3}{s+2} + \left[\frac{1}{2}\left(\frac{1}{s}\right) - 1\left(\frac{1}{s+1}\right) + \frac{1}{2}\left(\frac{1}{s+2}\right)\right][e^{-s} - e^{-2s}]$$

$$= \frac{2}{s+1} - \frac{1}{s+2} + \left[\frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{s+1} + \frac{1}{2}\left(\frac{1}{s+2}\right)\right][e^{-s} - e^{-2s}]$$

Thus

$$\boxed{\begin{aligned} y(t) = 2e^{-t} - e^{-2t} &+ \left[\tfrac{1}{2} - e^{-(t-1)} + \tfrac{1}{2}e^{-2(t-1)}\right]u_1(t) \\ &- \left[\tfrac{1}{2} - e^{-(t-2)} + \tfrac{1}{2}e^{-2(t-2)}\right]u_2(t) \end{aligned}}$$