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(1) The characteristic equation is:

$$r^2+2r+2=0 \Rightarrow r = -\frac{2}{2} \pm \frac{\sqrt{4-8}}{2} = -1 \pm i,$$

so the general solution to the homogeneous equation is:

$$y_h(x) = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t.$$

A particular solution to the non-homogeneous equation is of the form: $y_p = A\sin t + B\cos t$. We have

	$\sin t$	$\cos t$
$2y$	$2A$	$2B$
$2y'$	$-2B$	$2A$
y''	$-A$	$-B$

so. $y_p'' + 2y_p' + 2y_p =$

$(A-2B)\sin(t) + (2A+B)\cos t = 5\sin t.$

This gives $A - 2B = 5$

$2\times(2A + B = 0)$

$5A = 5 \Rightarrow \boxed{A=1} \Rightarrow -2B = 4 \Rightarrow \boxed{B=-2}$

$\boxed{y_p(t) = \sin(t) - 2\cos(t)}$.

Thus, $y_g(t) = \sin(t) - 2\cos(t) + c_1 e^{-t}\cos t + c_2 e^{-t}\sin(t)$.

$y(0) = -2 + c_1 = -2 \Rightarrow \boxed{c_1 = 0}$.

$y'(t) = \cos(t) - 2\sin(t) - c_2 e^{-t}\sin(t) + c_2 e^{-t}\cos(t)$

$y'(0) = 1 + c_2 = 1 \Rightarrow \boxed{c_2 = 0}$.

$\boxed{\therefore\ y(t) = \sin(t) - 2\cos(t)}$

2 $R = \sqrt{3^2 + (3\sqrt{3})^2} = \sqrt{9+27} = \sqrt{36} = 6$

$\omega = 2 \qquad \tan(\delta) = \dfrac{3\sqrt{3}}{3} = \sqrt{3} \Rightarrow \delta = \pi/3$

$\therefore$ (a) $\boxed{y(t) = 6\cos(2t - \pi/3)}$.

(b) The amplitude is 6, the natural frequency is 2, and the phase shift is $\pi/3$

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3 $xy = \sum_{n=1}^{\infty} a_{n-1}x^n$; $2y' = \sum_{n=0}^{\infty} 2(n+1)a_{n+1}x^n$; $y'' = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$

Thus, $2a_1 + 2a_2 = 0$; $\sum_{n=1}^{\infty}\left[a_{n-1} + 2(n+1)a_{n+1} + (n+2)(n+1)a_{n+2}\right]x^n = x$

We have $\boxed{a_0 = 1}$ $\boxed{a_1 = -1}$ $\boxed{a_2 = -a_1 = 1}$.

For $n=1$: $a_0 + 4a_2 + 6a_3 = 1$, $1 + 4 + 6a_3 = 1 \Rightarrow \boxed{a_3 = \frac{-2}{3}}$.

For $n=2$: $a_1 + 6a_3 + 12a_4 = 0$, $-1 \overset{-4}{\cancel{+6}} + 12a_4 = 0$

$\Rightarrow \boxed{a_4 = -5/12}$.

$\boxed{\therefore y(x) \approx 1 - x + x^2 - \frac{2}{3}x^3 + \frac{5}{12}x^4}$

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4. The Laplace transform of the equation is:

$s^2 Y - Y = \frac{1}{s}$, whence $Y = \frac{1}{s(s^2-1)}$.

A partial fraction decomposition is of the form:

$Y = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1} \Rightarrow A(s-1)(s+1) + B(s)(s+1) + Cs(s-1) = 1$,

~~whence~~ Setting $s=0$ gives $\boxed{A=-1}$.

Setting $s=1$ gives $\boxed{B=\frac{1}{2}}$. Setting $s=-1$ gives $\boxed{C=\frac{1}{2}}$

Thus $\mathcal{L}(y) = Y = \frac{-1}{s} + \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1}$,

so $\boxed{y(t) = -1 + \frac{1}{2}e^{t} + \frac{1}{2}e^{-t}}$.