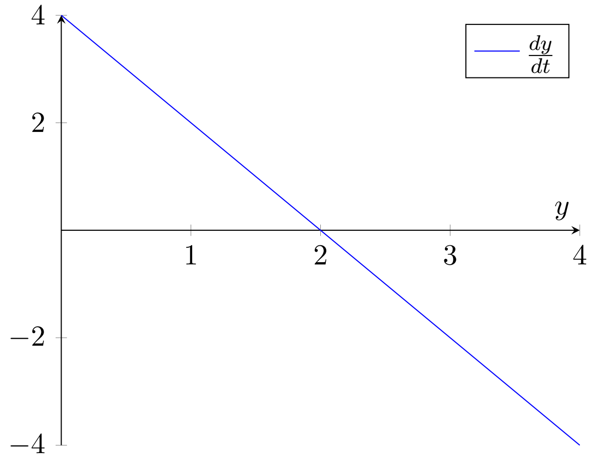
1. Consider $y' + 2y = 4$.

   (a) Does this equation have an equilibrium solution? If so, what is it?

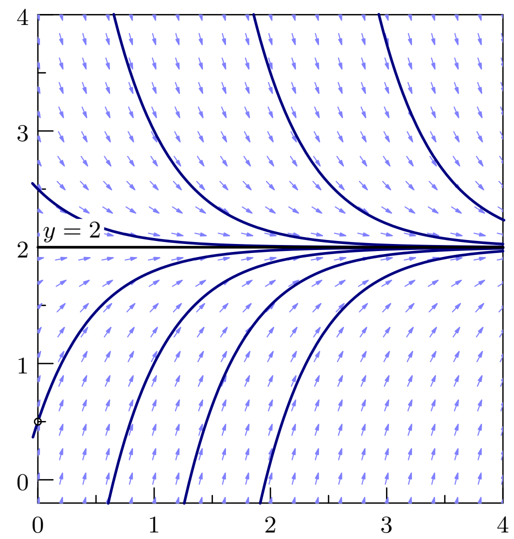
   Solution. Equilibrium will occur when $y' = 0$, that is when $\frac{dy}{dt}$ (the change in y over time) is nonexistent. Thus we want to solve $y' = 0 = -2y + 4$ for y. Doing so gives us an equilibrium solution of $y = 2$.

   (b) If there is an equilibrium solution, is it stable, unstable, or neither? State why.

   Solution. The equilibrium is stable because $y' < 0$ whenever $y > 2$ (i.e. when we start above equilibrium y is decreasing towards it) and $y' > 0$ whenever $y < 2$ (i.e when we start below equilibrium y is increasing towards it).



   (c) Sketch a direction field for this equation, showing any equilibrium solutions.



2. A certain small bay contains 83×10^6 ft^3 of water, and normally has a salinity of about 1.6%, corresponding to a concentration of about $1\frac{\text{lb}}{\text{ft}^3}$. In an exceptionally dry year, no water flowed into the bay over the summer, and the salinity rose to that of the open ocean, namely, about 3.5%, or about $2.2\frac{\text{lb}}{\text{ft}^3}$. With a return of rains, the river started discharging fresh water at the rate of $10.4 \times 10^6 \frac{\text{ft}^3}{\text{day}}$, and the fresh watter mixes with the bay watter. The bay water then flows into the ocean at the same rate. Biologists have determined that certain brackish-water fish will die unless the salinity returns to 2.5% ($< 1.5\frac{\text{lb}}{\text{ft}^3}$) within a week.

   (a) Write down an initial value problem that models the total amount of salt $S(t)$ in the bay at time t days after the river started flowing.

   Solution. We need to find both our initial condition (in the correct units) and S' in terms of S:

$$\begin{aligned} S(0) &= \left(83 \times 10^6 \text{ ft}^3\right)\left(2.2\frac{\text{lb}}{\text{ft}^3}\right) = 182.6 \times 10^6 \\ S' &= [\text{Rate in}] - [\text{Rate out}] \cdot [\text{Rate in}] = 0\frac{\text{lb}}{\text{day}} \\ [\text{Rate out}] &= \left(10.4 \times 10^6\right)\left(\frac{S}{83 \times 10^6}\right)\frac{\text{lb}}{\text{ft}^3} \\ &\approx .125S \Rightarrow S' \approx -.125S. \end{aligned}$$

   (b) Solve the initial value problem.

   Solution. Proceeding by separation of variables:

$$\begin{aligned} S' \approx -.125S &\iff \int \frac{dS}{S} \approx -.125 \int dt \\ &\iff \ln|S| \approx -.125t + k_1 \\ &\iff |S| \approx e^{((-.125t)+k_1)} = ke^{-.125t} \\ &\iff S(t) \approx 182.6 \times 10^6 e^{-.125t} \quad (\dagger) \end{aligned}$$

   ($\dagger$) since our initial condition is $S(0) = 182.6 \times 10^6$.

   (c) Compute the total amount of salt in the bay at time $t = 7$.

   Solution.

$$S(7) \approx 182.6 \times 10^6 e^{-.125t} \approx 76.1 \times 10^6 \text{ lbs}$$

   of salt after one week.

   (d) Will the salinity in the bay be less than 2.5% after one week?

   Solution.

$$\frac{76.1 \times 10^6 \text{ lbs}}{83 \times 10^6 \text{ ft}^3} < 1\frac{\text{lb}}{\text{ft}^3} < 1.5\frac{\text{lb}}{\text{ft}^3}.$$

   So, Yes the salinity will below the desired threshold.

3. Solve the following initial value problem: $y' - \frac{3}{t}y = t^3, y(1) = 1$.

Solution. This is a nonseprable (yet linear) differential equation. We therefore proceed by the technique of integrating factors to rewrite our equation (and make our life easier). To find our integrating factor we take the coefficient in front of y (this is a consequence of the mechanics of the power rule):

$$\mu = e^{\left(-3\int dt/t\right)} = e^{(-3\ln|t|)} = e^{\ln t^{-3}} = t^{-3}.$$

The general form for integrating factors is:

$$\mu y' + \mu\alpha y = \mu\beta.$$

In our specific case we have:

$$\begin{aligned}(t^{-3})y' + (t^{-3})(-3t^{-1})y = (t^{-3})t^3 &\iff t^{-3}y' - 3t^{-4} = 1\\ &\iff \frac{d}{dt}(t^{-3}y) = 1\\ &\iff \int\left[\frac{d}{dt}(t^{-3}y)\right]dt = \int dt\\ &\iff t^{-3}y = t + k\\ &\iff y = t^4 + kt^3.\end{aligned}$$

We now satisfy the given initial condition:

$$\begin{aligned}y(1) &= 1^4 + k\cdot 1^3\\ &= 1 + k = 1\\ &\Rightarrow k = 0\\ &\Rightarrow y = t^4.\end{aligned}$$

4. Solve the following initial value problem:

$$\frac{dy}{dx} = \sin(x)e^y, y(0) = 0.$$

Solution. We proceed via the method of separating variables.

$$\begin{aligned}\frac{dy}{dx} = \sin(x)e^y &\iff dy = (sin(x)e^y)\,dx\\ &\iff \int e^{-y}dy = \int sin(x)dx\\ &\iff -e^{-y} = -\cos(x) + k\\ &\iff \ln(e^{-y}) = \ln[\cos(x) + k]\\ &\iff y = -\ln[\cos(x) + k].\end{aligned}$$

Now to satisfy our initial condition:

$$\begin{aligned}y(0) &= -\ln(\cos(0) + k)\\ &= -\ln(1 + k) = 0\\ &\Leftrightarrow e^{[\ln(1+k)]} = e^0\\ &\Leftrightarrow 1 + k = 1\\ &\Rightarrow k = 0\\ &\Rightarrow y = -\ln\left[\cos x\right].\end{aligned}$$