(1) $r^2+4r+5=0 \Rightarrow r = \frac{-4 \pm \sqrt{4^2-4(1)(5)}}{2} = -2 \pm i$

This gives the general solution:

$$\boxed{y(t) = e^{-2t}[C_1\cos(t)+C_2\sin(t)]}$$

The solutions tend to 0 exponentially as $t \to \infty$.

(2) The homogeneous equation has characteristic equation: $r^2+5r+4=0$, giving $r=-1$ and $r=-4$.

The solution is thus:

$$y_h(t) = C_1e^{-t}+C_2e^{-4t}.$$

We assume $y_p(t) = A\cos(t)+B\sin(t)$.

$$y_p' = -A\sin(t)+B\cos(t)$$

$$y_p'' = -A\cos(t)-B\sin(t)$$

so $y_p''+5y_p'+4y_p = (-A+5B+4A)\cos t + (-B-5A+4B)\sin(t)$

$= (3A+5B)\cos(t)+(-5A+3B)\sin(t)$

$= (0)\cos(t)+(1)\sin(t)$,

giving $3A+5B=0$, $-5A+3B=1$, whence

$$\begin{aligned} 15A+25B&=0\\ -15A+9B&=3\\ \hline 34B&=3 \Rightarrow B=\frac{3}{34} \end{aligned}$$

$$A = \frac{-5\left(\frac{3}{34}\right)}{3} = \frac{-5}{34}.$$

$$\boxed{\therefore y_g(t) = C_1e^{-t}+C_2e^{-4t}-\frac{5}{34}\cos(t)+\frac{3}{34}\sin(t)}$$

3. $r^2+10r+25=0 \Rightarrow r=-5$.

Thus, $y_g(t)=c_1e^{-5t}+c_2te^{-5t}$.

This gives $y(0)=c_1=0$

$y'(t)=\cancel{-5c_1e^{-5t}}\ -5c_1e^{-5t}+c_2e^{-5t}-5c_2te^{-5t}$

$\Rightarrow y'(0)=c_2=1$.

$$\boxed{\therefore\ y(t)=te^{-5t}}$$

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4. From problem 2,

$y_h(t)=c_1e^{-t}+c_2e^{-4t}$. The particular solution must therefore be of the form

$y_p(t)=A\,te^{-t}$

$y_p'=Ae^{-t}-Ate^{-t}$

$y_p''=-2Ae^{-t}+Ate^{-t}$

so $y_p''+5y_p'+4y_p=(A-5A+4A)te^{-t}+(-2A+5A)e^{-t}$

$=3Ae^{-t}=e^{-t}$

$\Rightarrow 3A=1 \Rightarrow A=1/3$.

Thus, the general solution is:

$$\boxed{y_g(t)=\tfrac{1}{3}te^{-t}+c_1e^{-t}+c_2e^{-4t}}$$