$$y'' + 3y' + 2y = u_2(t)e^{t-2},\ y(0) = 0,\ y'(0) = 1.$$

$$\mathcal{L}(y'') + 3\mathcal{L}(y') + 2\mathcal{L}(y) = \mathcal{L}(u_2(t)e^{t})$$

$$s^2Y - sy(0) - y'(0) + 3sY - 3y(0) + 2Y = e^{-2s}\cdot\left[\frac{1}{s-1}\right].$$

$$(s^2+3s+2)Y - 1 = e^{-2s}\cdot\left[\frac{1}{s-1}\right].$$

$$\Rightarrow Y(s) = \frac{1}{s^2+3s+2} + e^{-2s}\cdot\left[\frac{1}{(s-1)(s^2+3s+2)}\right].$$

We now need to do two partial fraction decompositions:

$$\frac{1}{(s^2+3s+2)} = \frac{A}{s+1} + \frac{B}{s+2} \Rightarrow 1 = A(s+2) + B(s+1)$$
$$= (A+B)s + (2A+B),$$

$\Rightarrow A+B=0,\ 2A+B=1.\ B=-A$, so $2A-A=1$, or $A=1,\ B=-1.$

$$\boxed{\frac{1}{s^2+3s+2} = \frac{1}{s+1} - \frac{1}{s+2}.}$$

$$\frac{1}{(s-1)(s^2+3s+2)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+2}.$$

$$1 = A(s^2+3s+2) + B(s^2+s-2) + C(s^2-1)$$
$$= (A+B+C)s^2 + (3A+B)s + (2A-2B-C)$$

$\Rightarrow$

$$A+B+C = 0 \ \text{(i)}$$
$$3A+B = 0 \ \text{(ii)}$$
$$2A-2B-C = 1 \ \text{(iii)}$$

adding (i) and (iii) gives $3A-B=1$.
adding this to (ii) gives $6A=1 \Rightarrow A=1/6$.
Plugging this into (ii) gives $B = -3/6 = -1/2$
Plugging these values into (i) gives $C = -\frac{1}{6} - \left(\frac{-1}{2}\right) = \frac{1}{3}.$

$$\boxed{\frac{1}{(s-1)(s^2+3s+2)} = \frac{+1/6}{(s-1)} + \frac{-1/2}{s+1} + \frac{1/3}{s+2}.}$$

thus,

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) + \frac{1}{6}\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s-1}\right) - \frac{1}{2}\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s+1}\right) + \frac{1}{3}\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s+2}\right).$$

$$= \boxed{e^{-t} - e^{-2t} + \frac{1}{6}u_2(t)e^{+(t-2)} - \frac{1}{2}u_2(t)e^{-(t-2)} + \frac{1}{3}u_2(t)e^{-2(t-2)}}$$