1 $y=\sum_{n=0}^{\infty} a_n x^n$; $y'=\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{m=0}^{\infty} (m+1)a_{m+1}x^m$

$$y''=\sum_{n=2}^{\infty} (n)(n-1)a_n x^{n-2} = \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m.$$

Thus, $y''-y' = \sum_{m=0}^{\infty} [(m+2)(m+1)a_{m+2}-(m+1)a_{m+1}]x^m = \sum_{m=0}^{\infty} \frac{1}{m!}x^m$

Thus $(m+2)(m+1)a_{m+2}-(m+1)a_{m+1}=\frac{1}{m!}, \ m\geq 0.$

That is, $a_{m+2} = \dfrac{\frac{1}{m!}+(m+1)a_{m+1}}{(m+2)(m+1)} = \dfrac{1}{(m+2)!}+\dfrac{a_{m+1}}{(m+2)}, \ m\geq 0.$

$a_0=0, \ a_1=1, \ a_2=\frac{1}{2!}+\frac{a_1}{2}=\frac{1}{2}+\frac{1}{2}=1.$

$a_3=\frac{1}{3!}+\frac{a_2}{3}=\frac{1}{6}+\frac{1}{3}=\frac{1}{2}=\frac{1}{2!}$

$a_4=\frac{1}{4!}+\frac{a_3}{4}=\frac{1}{24}+\frac{1}{8}=\frac{1}{6}=\frac{1}{3!}$

$a_5=\frac{1}{5!}+\frac{a_4}{5}=\frac{1}{120}+\frac{1}{30}=\frac{5}{120}=\frac{1}{24}=\frac{1}{4!}.$

Thus, the degree-5 Taylor approximation to the solution is: $\boxed{x+x^2+\frac{x^3}{2!}+\frac{x^4}{3!}+\frac{x^5}{4!}}$.

$$y(x) = \sum_{n=0}^{\infty} a_n (x+1)^n; \quad y'(x) = \sum_{n=1}^{\infty} n a_n (x+1)^{n-1}.$$

$$xy' = (x+1)y' - y' = \sum_{n=1}^{\infty} n a_n (x+1)^n - \sum_{n=1}^{\infty} n a_n (x+1)^{n-1}$$

$$= \sum_{n=1}^{\infty} n a_n (x+1)^n - \sum_{m=0}^{\infty} (m+1) a_{m+1} (x+1)^m.$$

Thus, $xy' - 2y = \sum_{n=1}^{\infty} [n a_n - (n+1) a_{n+1} - 2a_n](x+1)^n$

$$-2a_0 - a_1 = 0$$

$$a_0 = 1, \quad a_1 = -2a_0 = -2.$$

For $n \geq 1$: $(n-2)a_n = (n+1)a_{n+1}$

$$\Rightarrow a_{n+1} = \frac{(n-2)a_n}{n+1}. \quad a_2 = \frac{-a_1}{2} = 1.$$

$$a_3 = \frac{0 a_2}{3} = 0 = a_4 = a_5 = \cdots$$

Thus, the exact solution is:

$$\boxed{y(x) = 1 - 2(x+1) + (x+1)^2}.$$

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(3) $\mathcal{L}(y''-y') = \mathcal{L}(y'') - \mathcal{L}(y') = s^2Y - sy(0) - y'(0) - sY - y(0)$

$= \mathcal{L}(e^x) = \frac{1}{s-1}.$

Plugging in $y(0)=0, y'(0)=1$ gives:

$(s^2-s)Y - 1 = \frac{1}{s-1} \Rightarrow Y = \frac{1}{s(s-1)^2} + \frac{1}{s(s-1)}.$

We now need to do a partial fraction decomposition for each term: $\frac{1}{s(s-1)^2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$

$1 = A(s-1)^2 + Bs(s-1) + Cs$

$1 = (A+B)s^2 + (-2A-B+C)s + A$

This gives $A=1, B=-1$; $-2(1)-(-1)+C=0 \Rightarrow C=1$

so $\frac{1}{s(s-1)^2} = \frac{1}{s} - \frac{1}{s-1} + \frac{1}{(s-1)^2}.$

$\frac{1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1}$, or $1 = A(s-1) + Bs = (A+B)s - A$

$\Rightarrow A=-1, B=1$, i.e. $\frac{1}{s(s-1)} = \frac{-1}{s} + \frac{1}{s-1}.$

Combining gives:

$Y = \frac{1}{s} - \frac{1}{s-1} + \frac{1}{(s-1)^2} - \frac{1}{s} + \frac{1}{s-1} = \frac{1}{(s-1)^2},$

so $y(x) = \mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right) = \boxed{xe^x \text{ by formula 11 in the table}}.$

(4) The characteristic equation is:

$\alpha^2 - \alpha = 0$, which has roots $\alpha = 0$, $\alpha = 1$, so the solution to the corresponding homogeneous equation is:

$$y_h(x) = C_1 + C_2 e^x.$$

Since the RHS is a solution to the homogeneous equation, we need to assume a solution of the form $Axe^x = y$.

Plugging in $y' = A(e^x + xe^x)$; $y'' = A(2e^x + xe^x)$:

$$y'' - y' = 2Ae^x + Axe^x - Ae^x - Axe^x$$
$$= Ae^x = e^x \Rightarrow A = 1.$$

Thus, the general solution to the nonhomogeneous equation is:

$$y(x) = C_1 + C_2 e^x + xe^x.$$

$$y(0) = C_1 + C_2 = 0$$
$$y'(x) = C_2 + e^x + xe^x$$
$$y'(0) = C_2 + 1 = 1 \Rightarrow C_2 = 0.$$

Thus, the solution to the initial value problem is: $\boxed{y(x) = xe^x}$

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5 $y' + \frac{y}{x} = 2, \quad y(1) = 1$

an integrating factor is: $\mu(x) = e^{\int \frac{dx}{x}} = e^{\ln(x)} = x$

$xy' + y = 2x$

$(xy)' = 2x \Rightarrow xy = x^2 + C \Rightarrow y = x + \frac{c}{x}$

$y(1) = 1 + C = 1 \Rightarrow C = 0 \Rightarrow \boxed{y(x) = x}$