Math. 350-01, Wednesday, September 13, 2006, First Exam Answers

1 (a) This is linear, first-order.

(b) This is linear, second-order.

(c) This is nonlinear since the product yy' occurs. It is second order.

(d) This is nonlinear, since a $\sin(y)$ term occurs; it is second-order.

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2 $u(t) = e^{\int 2t\,dt} = e^{t^2}$, so $(e^{t^2}y)' = 2te^{t^2}$,

so $e^{t^2}y = \int 2te^{t^2}dt = e^{t^2} + C$, so $y(t) = 1 + Ce^{-t^2}$.

Thus,

$y(0) = 2 \Leftrightarrow 1 + C = 2 \Rightarrow C = 1 \Rightarrow \boxed{y(t) = 1 + e^{-t^2}}$.

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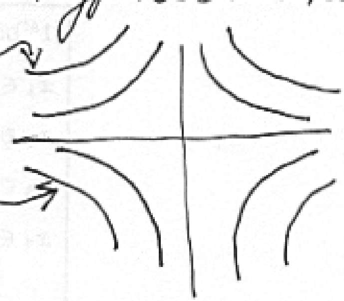
3 $\frac{dy}{dx} = -\frac{y}{x} \Leftrightarrow \frac{dy}{y} = \frac{-dx}{x} \Rightarrow \ln(|y|) = \ln\left(\frac{1}{|x|}\right) + C$

$\Rightarrow |y| = \frac{k}{|x|}$. The solution curves are hyperbolas skewed 45° with respect to the x-axis.

Solution curves



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4 (a) (i) $y' = -\alpha y$

(b) (ii) $y(t) = Ce^{-\alpha t}$

(iii) $y(5{,}730) = Ce^{-5730\alpha} = \frac{1}{2}y(0) = \frac{1}{2}C$.

This gives $e^{-5730\alpha} = \frac{1}{2} \Rightarrow -5730\alpha = \ln\left(\frac{1}{2}\right) = -\ln(2)$

$\Rightarrow \alpha = \frac{\ln(2)}{5730} \Rightarrow y(t) = Ce^{-\frac{\ln 2}{5730}t} = C\left(\frac{1}{2}\right)^{t/5730}$.

(iv) $y(0) = C = 100 \Rightarrow \boxed{y(t) = 100\left(\frac{1}{2}\right)^{t/5730}}$

~~3~~ (b) The age T obeys $94 = 100\left(\frac{1}{2}\right)^{T/5730}$, or

$\left(\frac{1}{2}\right)^{T/5730} = .94$. Thus, $\frac{-T}{5730}\ln(2) = \ln(.94)$

$$\Rightarrow \boxed{T = \frac{-5730\ln(.94)}{\ln(2)} \approx 511.5 \text{ years}}$$

(5a) $\frac{dx}{dt} = \alpha(p-x)^2$. To solve:

$$\int \frac{dx}{(p-x)^2} = \int \alpha\, dt \Leftrightarrow \frac{1}{p-x} = \alpha t + C$$

$$x(0)=0 \Rightarrow \frac{1}{p} = C \Rightarrow \frac{1}{p-x} = \alpha t + \frac{1}{p} \Rightarrow p-x = \frac{1}{\alpha t + \frac{1}{p}}$$

$$\Leftrightarrow p-x = \frac{p}{\alpha p t+1} \Rightarrow x = p\left[1 - \frac{1}{\alpha p t+1}\right].$$

(b) As $t \to \infty$, $x \to p$