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(1) a direction vector for the line is $\langle 4,5,6\rangle - \langle 1,2,3\rangle = \langle 3,3,3\rangle$.
Thus, an equation for the plane is:

$$\langle 3,3,3\rangle \cdot (\langle x,y,z\rangle - \langle 1,0,0\rangle) = 0, \text{ i.e.}$$

$$3(x-1) + 3(y-0) + 3(z-0) = 0, \text{ or } \boxed{x+y+z=1}.$$

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(2) $\frac{\partial f}{\partial x} = \cos(xy) - xy\sin(xy)$, so $\left.\frac{\partial f}{\partial x}\right|_{(1,\pi)} = \cos(\pi) = -1$.
$\frac{\partial f}{\partial y} = -x^2\sin(xy)$, so $\left.\frac{\partial f}{\partial y}\right|_{(1,\pi)} = 0$. Also, $f(1,\pi) = 0$.
Thus, an equation for the tangent line is:

$$z + \cancel{1} = \langle -1,0\rangle \cdot \langle x-1, y-\pi\rangle = -(x-1) = 1-x.$$

$$z \cancel{+1} = 1-x, \text{ or } \boxed{x+z=0}$$

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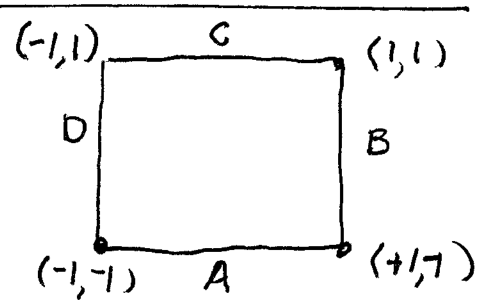
(3) We first find the critical points.



$\frac{\partial f}{\partial x} = x^2 - y^2 = 0$ at $y = \pm x$.

$\frac{\partial f}{\partial y} = -2xy + y = y(1-2x) = 0$ at $y=0$ or $x = 1/2$.

Thus, the critical points are $\boxed{(0,0), (\tfrac{1}{2}, -\tfrac{1}{2}), \text{ and } (\tfrac{1}{2}, \tfrac{1}{2})}$

On side A: $g(x) = f(x,-1) = \frac{1}{3}x^3 - x + \frac{3}{2}$, so $g'(x) = x^2 - 1 = 0$ at $x = \pm 1$, end points of the line segment. ~~It is~~ Similarly, the constrained critical points on side C also occur on the end points.

On side B: $h(y) = f(1,y) = \frac{1}{3} - y^2 + \frac{1}{2}y^2 + 1 = -\frac{1}{2}y^2 + \frac{4}{3}$
$h'(y) = -y = 0$ at $y = 0$, giving the point $\boxed{(1,0)}$.

On side D: $\tilde{h}(y) = \frac{-1}{3} + y^2 + \frac{1}{2}y^2 + 1$, so $\tilde{h}'(y) = 3y = 0$ at $y=0$, giving the point $(-1,0)$

⋮

3, continued:

Putting all of these together and including the corners gives this table of candidates.

Thus the maximum is $2\frac{1}{6}$, and it occurs at the corners $(-1,-1)$ and $(-1,1)$, while the minimum is $2/3$, occurring at $(-1,0)$

	(x,y)	$f(x,y)$
corners	$(-1,-1)$	$2\frac{1}{6}$
	$(1,-1)$	$5/6$
	$(1,1)$	$5/6$
	$(-1,1)$	$2\frac{1}{6}$
sides	$(1,0)$	$1\frac{1}{3}$
	$(-1,0)$	$2/3$
interior	$(0,0)$	1
	$(\frac{1}{2},-\frac{1}{2})$	$1\frac{1}{24}$
	$(\frac{1}{2},\frac{1}{2})$	$1\frac{1}{24}$

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(4) We will use cylindrical coordinates, with $0 \le r \le 1$, $0 \le \theta \le 2\pi$, $0 \le z \le 2$. We have

$$\iiint_{\mathcal{V}} x^2+y^2\,dV = \int_{z=0}^{2}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^2(r\,dr)\,d\theta\,dz$$

$$= \left\{\int_{z=0}^{2} dz\right\}\left\{\int_{\theta=0}^{2\pi} d\theta\right\}\left\{\int_{r=0}^{1} r^3\,dr\right\} = (2)(2\pi)\left(\frac{1}{4}\right) = \boxed{\pi}.$$

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(5) Note that we have a parametrization of the unit sphere, and, on that sphere, $\langle x,y,z\rangle$ is a unit vector pointing outward. Thus,

$$\iint_{S} \langle x,y,z\rangle\circ\vec{n}\,dS = \iint_{S} dS = \boxed{4\pi}.$$

(You can also use the divergence theorem.)

6 (a) $\frac{\partial F_1}{\partial y} = -2y = \frac{\partial F_2}{\partial x}$, so $\vec{F}$ is conservative.

(b) a potential function for $\vec{F}$ is:

$\varphi(x,y) = \frac{1}{3}x^3 - xy^2 + g(y)$; $\frac{\partial \varphi}{\partial y} = -2xy + g'(y) = -2xy + y$,

so $g'(y) = y$, so $g(y) = \frac{1}{2}y^2 + C$.

Thus, $\boxed{\varphi(x,y) = \frac{1}{3}x^3 - xy^2 + \frac{1}{2}y^2 + C}$.

Therefore, $\int_{\mathcal{C}} \vec{F} \circ d\vec{r} = \varphi(1,1) - \varphi(0,0) = \frac{1}{3} - 1 + \frac{1}{2} = \boxed{\frac{-1}{6}}$.

7 $\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{0}$, so

$\iint_{\mathcal{S}} \operatorname{curl}(\vec{F}) \cdot \vec{n}\, dS = 0$ for any surface bounded by the unit circle $\mathcal{C}$. Therefore, by Stokes' theorem,

$\int_{\mathcal{C}} \vec{F} \circ d\vec{r} = 0$.