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(1) $\frac{\partial f}{\partial x} = 2x+2y-4$, $\frac{\partial f}{\partial y} = 2x+4y-6$

Solving $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$:
$$-[2x+2y=4]$$
$$2x+4y=6$$
$$2y=2 \Rightarrow \boxed{y=1.}$$

$2x+2=4 \Rightarrow \boxed{x=1}$.

$\boxed{\therefore (x,y)=(1,1)}$ is the only critical point.

We compute the second derivatives to classify it —

$$\frac{\partial^2 f}{\partial x^2}=2,\ \frac{\partial^2 f}{\partial y^2}=4,\ \frac{\partial^2 f}{\partial x \partial y}=2.$$

At $(x,y)=(1,1)$, $\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2=(2)(4)-4=4>0$.

Therefore, $(x,y)=(1,1)$ corresponds to a local minimum (since $\frac{\partial^2 f}{\partial x^2}>0$).

(2a) $\nabla f = \begin{bmatrix} 2x+2y-4 \\ 2x+4y-6 \end{bmatrix}$. $\nabla g = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\nabla f = \lambda \nabla g \Leftrightarrow \left| \begin{array}{l} 2x+2y-4=\lambda \\ 2x+4y-6=\lambda \end{array}\right. \Rightarrow \cancel{2x}+2y-4=\cancel{2x}+4y-6$

Plugging into the constraint $\Rightarrow 2y=2 \Rightarrow \boxed{y=1}$.

now gives $\boxed{x=0}$.

Thus the minimum is at $(0,1)$, and its value is ~~at~~ 1.

To see that this is actually a minimum, observe what happens as x and y get large, or proceed to part (b)

(2b) $y=1-x$, so $\tilde{f}(x)=f(x,1-x)=x^2+2(1-x)x-4x+2(1-x)^2-6(1-x)+5$.

$\tilde{f}'(x)=2x+2-4x-4\cancel{x}-4(1-x)+6=\cancel{2x}\ 2x+0=0 \Rightarrow \boxed{x=0, y=1}$.

Since $\tilde{f}$ is a quadratic and its leading term is positive, it has a single global minimum.

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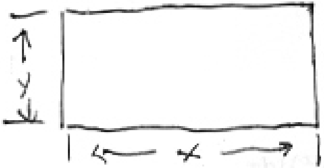

The problem is max xy

subject to $2(2x+y)=250$

i.e. $4x+2y=250$.

$f(x,y)=xy$ $\quad \nabla f = \begin{bmatrix} y \\ x \end{bmatrix}$ $\quad \nabla g = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$g(x,y)=4x+2y$

$\nabla f = \lambda \nabla g \Rightarrow \left.\begin{matrix} y=4\lambda \\ x=2\lambda \end{matrix}\right\} \Rightarrow y=2x$. Plugging this

into the constraint gives: $4x+2(2x)=250 \Rightarrow 8x=250$

$\Rightarrow x=31.25$ meters, $y=62.5$ meters.