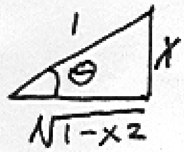
Math. 301-01, Spring, 2007, Second Exam Answers, page 1

(1)



$x = \sin(\theta)$
$dx = \cos(\theta)\,d\theta$
$\sqrt{1-x^2} = \cos\theta.$

$$\int_0^1 \sqrt{1-x^2}\,dx = \int_{\theta=\arc\cancel{\cos}\sin(0)}^{\theta=\arcsin(1)} \cos(\theta)(\cos(\theta)\,d\theta = \int_{\theta=0}^{\pi/2} \cos^2(\theta)\,d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[\frac{1}{2}+\frac{1}{2}\cos(2\theta)\right]d\theta = \left[\frac{\theta}{2}+\frac{1}{4}\sin(2\theta)\right]\Bigg|_{\theta=0}^{\pi/2}$$

$= \left[\frac{\pi}{4}+0\right]-\left[0+0\right] = \boxed{\pi/4}$. alternately, one can see that the integral is $\frac{1}{4}$ the area of a circle of radius 1, that is, $\frac{1}{4}(\pi\cdot 1^2) = \boxed{\frac{\pi}{4}}$.

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(2) $$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos(\theta)\,d\theta}{\cos(\theta)} = \int d\theta$$ ← Use the same substitution as in problem 1

$= \theta + C = \boxed{\arcsin(x)+C}$

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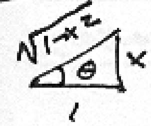
(3) $$\int \frac{2x}{\sqrt{1-x^2}}\,dx = -\int \frac{du}{\sqrt{u}}$$

$u = 1-x^2$
$-du = 2x\,dx$

$= -2u^{1/2}+C$
$= \boxed{-2\sqrt{1-x^2}+C}.$

alternately:

$x = \sin(\theta)$
$dx = \cos(\theta)\,d\theta$
$\cos(\theta) = \sqrt{1-x^2}$



so $$\int \frac{2x}{\sqrt{1-x^2}}\,dx = \int \frac{2\sin\theta\cos\theta\,d\theta}{\cos(\theta)} = 2\int \sin(\theta)\,d\theta$$

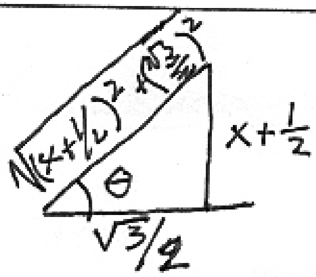
$= -2\cos(\theta)+C = \boxed{-2\sqrt{1-x^2}+C}$

(4) $$\int \frac{x+1}{x-1}\,dx = \int 1 + \frac{2}{x-1}\,dx$$

$$= \boxed{x + 2\ln|x-1| + C}$$

$$\leftarrow \begin{array}{r|l} & \ \ 1 \\ x-1 & \overline{x+1} \\ & \underline{x-1} \\ & \ \ \ 2 \end{array}$$

(5) $$\int \frac{dx}{x^2+x+1} = \int \frac{dx}{(x+1/2)^2 - 1/4 + 1} = \int \frac{dx}{(x+1/2)^2 + 3/4}$$



$$\tan\theta = \frac{x+1/2}{\sqrt{3}/2}$$

$$\sec^2\theta\, d\theta = \frac{2}{\sqrt{3}}\,dx$$

$$\frac{\sqrt{3}}{2}\sec(\theta) = \sqrt{(x+1/2)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \int \frac{\sqrt{3}/2\,\sec^2\theta\,d\theta}{3/4\,\sec^2\theta} = \frac{2}{\sqrt{3}}\int d\theta$$

$$= \frac{2}{\sqrt{3}}\theta + C = \boxed{\frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}(x+1/2)\right) + C}$$

(6) Use formula 17: $\int \sin^3 x\,dx = -\frac{1}{3}\sin^2 x\cos x + \frac{2}{3}\int \sin(x)\,dx$

$$= \frac{1}{3}\sin^2 x\cos x - \frac{2}{3}\cos(x) + C,$$

so
$$\int_0^{\pi/2} \sin^3(x)\,dx = \left[\frac{1}{3}\sin^2\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) - \frac{2}{3}\cos\left(\frac{\pi}{2}\right)\right]$$

$$- \left[\frac{1}{3}\sin^2(0)\cos(0) - \frac{2}{3}\cos(0)\right]$$

$$= [\ 0 \ - 0\ ] - \left[0 - \frac{2}{3}\right]$$

$$= \boxed{\frac{2}{3}}$$